A special family of K3 surfaces with hypergeometric properties

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Invertible Pencils

Let $A=(a_{ij})_{i,j}$ is a $(n+1)\times(n+1)$ matrix of natural numbers. Define:

$$F_A := \sum_{i=0}^{n}\prod_{j=0}^{n} x_j^{a_{ij}} \in \mathbb{Z}[x_0,\dots,x_n].$$

A *homogeneous invertible polynomial* is a polynomial of the form F_A satisfying:

1. $\det(A)\neq 0$.
2. F_A is homogeneous of degree $n+1$.
3. $F_A : \mathbb{C}^{n+1}\to\mathbb{C}$ has a unique critical point at the origin.

For a parameter ψ, we can deform an invertible polynomial and to get a family of polynomials:

$$F_A - d^T\psi x_0\cdots x_n,$$

where $d^T\in\mathbb{Z}$.

The family of varieties defined by those polynomials

$$V(F_A - d^T\psi x_0\cdots x_n)\subset\mathbb{P}^n$$

is called an *invertible pencil*.

Classification

There is a classification of invertible polynomials (and consequently pencils) due to Kreuzer and Skarke [KS92]. For $n = 3$ it has ten classes

1. $x^4 + y^4 + z^4 + w^4 - 4\psi xyzw$
2. $x^3y + y^3z + z^3x + w^4 - 4\psi xyzw$
3. $x^3y + y^3x + z^4 + w^4 - 4\psi xyzw$
4. $x^3y + y^3x + z^3w + w^3z - 4\psi xyzw$
5. $x^3y + y^3z + z^3w + w^3x - 4\psi xyzw$
6. $x^3y + y^4 + z^4 + w^4 - 12\psi xyzw$
7. $x^3y + y^4 + z^3w + w^3z - 12\psi xyzw$
8. $x^3y + y^4 + z^3w + w^4 - 6\psi xyzw$
9. $x^3y + y^3z + z^4 + w^4 - 36\psi xyzw$
10. $x^3y + y^3z + z^3w + w^4 - 27\psi xyzw$

Those pencils define families of **quartic hypersurfaces in** $\mathbb{P}^3$ whose generic member is smooth.

K3 surfaces

Definition

A K3 surface over a field k (for today, think $\mathbb{C}$ or $\mathbb{F}_q$) is a smooth irreducible algebraic variety X of dimension two such that $\omega_X := \Omega^2_{X/k} \simeq \mathscr{O}_X$ and $H^1(X, \mathscr{O}_X) = 0$.

Example

Every **smooth quartic hypersurface** $V(f) \subset \mathbb{P}^3$, **where** $f \in \Gamma(\mathbb{P}^3, \mathscr{O}(4))$ is a degree four homogeneous polynomial, is a K3 surface.

We can think of K3 surfaces as a generalisation of elliptic curves (rich arithmetic) in dimension two.

Goal

The pencils on the left were studied in [Dor+20]. Precisely, the authors computed the L-functions of each pencil using hypergeometric functions.

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Our goal is to understand the arithmetic (L-functions) of the remaining five invertible pencils:

6. $x^3y + y^4 + z^4 + w^4 - 12\psi xyzw$
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Today we focus on

$$X_\psi : x^3y + y^3z + z^3w + w^4 - 27\psi xyzw = 0.$$

Varieties over finite fields and point counts

Let p be a prime and q a power of p. If X is a variety defined over $\mathbb{F}_q$, given a finite extension $\mathbb{F}_{q^m}$, we can compute $\#X(\mathbb{F}_{q^m})$ for every $m \geq 0$. We then pack them together in the zeta function

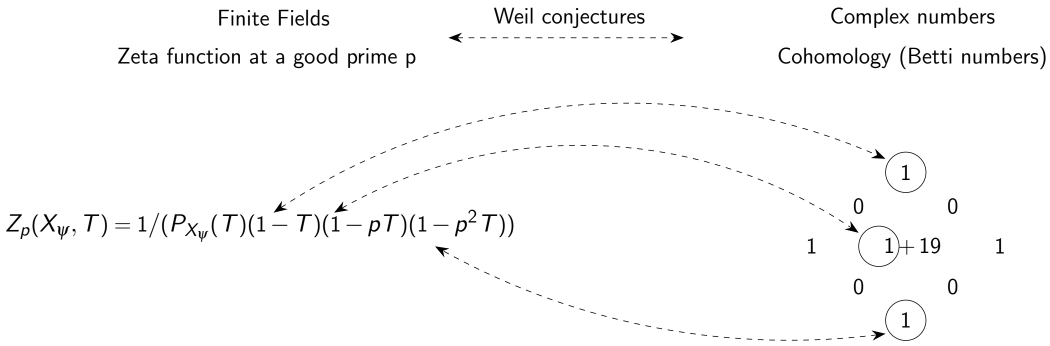
$$Z_q(X,T) = \exp\left(\sum_{m=1}^{\infty} \#X(\mathbb{F}_{q^m})\frac{T^m}{m}\right).$$

For our variety X_ψ, the set $X_\psi(\mathbb{F}_{q^m})$ will be the set of points of $\mathbb{P}^3_{\mathbb{F}_{q^m}}$ satisfying the equation

$$x^3y + y^3z + z^3w + w^4 - 27\psi xyzw = 0. \tag{1}$$

Thus, $\#X_\psi(\mathbb{F}_{q^m})$ "counts solutions for equation (1) with coordinates in $\mathbb{F}_{q^m}$".

Weil conjectures in the context of smooth quartics in $\mathbb{P}^3$



Finite Fields

Zeta function at a good prime p

Weil conjectures

Complex numbers

Cohomology (Betti numbers)

$$Z_p(X_\psi, T) = 1/(P_{X_\psi}(T)(1-T)(1-pT)(1-p^2T))$$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 1+19 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Let $S = S(\psi)$ denote the set of good primes of X_ψ, we define

$$L_S(X_\psi, s) := \prod_{p \notin S} (P_{X_\psi}(p^{-s}))^{-1}.$$

Hypergeometrics over $\mathbb{C}$

If $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_d\}, \boldsymbol{\beta} = \{\beta_1, \ldots, \beta_d\} \subset \mathbb{Q}$ are d-multisets, the *generalised hypergeometric series* associated to $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is defined as

$$ {}_dF_{d-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_d)_n} x^n, $$

where $(a)_n = \begin{cases} a(a+1)\cdots(a+n-1) \text{ if } n > 0 \\ 1, \text{ if } n = 0 \end{cases}$.

If $1 \in \boldsymbol{\beta}$, then ${}_dF_{d-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid x)$ converges for $|x| < 1$.

Hypergeometric functions over $\mathbb{C}$ are related to periods of invertible pencils (right-hand side of the previous diagram).

Hypergeometrics over $\mathbb{F}_q$

Let $q^\times := q-1$.

- We would like to define a $\mathbb{F}_q$-version of the hypergeometric series.
- We start by rewriting it in terms of the Γ function

$$ {}_dF_{d-1}(\boldsymbol{\alpha},\boldsymbol{\beta} \mid z) = \sum_{n=0}^{\infty} \prod_{i=1}^{d} \frac{\Gamma(\alpha_i+n)\Gamma(1-n-\beta_i)}{\Gamma(\alpha_i)\Gamma(1-\beta_i)}(-1)^{nd}z^n. $$

- We recall that Gauss sums are finite–field analogues of Γ.
- We take ω a generator of the group $\mathrm{Hom}(\mathbb{F}_q^\times,\mathbb{C}^\times)$, Θ non-trivial element of $\mathrm{Hom}((\mathbb{F}_q,+),\mathbb{C}^\times)$, $m\in\mathbb{Z}$ and define the *Gauss sum* as

$$ g(m) = \sum_{x\in\mathbb{F}_q^\times} \omega^m(x)\Theta(x) \in \mathbb{C}. $$

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$$ {}_dF_{d-1}(\boldsymbol{\alpha},\boldsymbol{\beta} \mid z) = \sum_{n=0}^{\infty} \prod_{i=1}^{d} \frac{\Gamma(\alpha_i+n)\Gamma(1-n-\beta_i)}{\Gamma(\alpha_i)\Gamma(1-\beta_i)} (-1)^{nd} z^n. $$

Definition (McCarthy,Katz,Greene)

Now suppose that $\boldsymbol{\alpha},\boldsymbol{\beta}$ satisfy $q^\times\alpha_i, q^\times\beta_i \in \mathbb{Z}$, $i=1,\dots,d$. For $t \in \mathbb{F}_q^\times$ we define

$$ H_q(\boldsymbol{\alpha},\boldsymbol{\beta} \mid t) = -\frac{1}{q^\times} \sum_{m=0}^{q-2} \prod_{i=1}^{d} \frac{g(m+\alpha_i q^\times)g(-m-\beta_i q^\times)}{g(\alpha_i q^\times)g(-\beta_i q^\times)} \omega((-1)^d t)^m. $$

Warning!

The conditions $q^\times\alpha_i, q^\times\beta_i \in \mathbb{Z}$ for every $i=1,\dots,d$ are restrictive. There is a way to remove them and give a more general definition. I won't talk about it.

Let

$$X_\psi : x^3y + y^3z + z^3w + w^4 - 27\psi xyzw = 0$$

and consider the parameters

$$\begin{aligned}\boldsymbol{\alpha} &= \left\{\tfrac{1}{27}, \tfrac{2}{27}, \tfrac{4}{27}, \tfrac{5}{27}, \tfrac{7}{27}, \tfrac{8}{27}, \tfrac{10}{27}, \tfrac{11}{27}, \tfrac{13}{27}, \tfrac{14}{27}, \tfrac{16}{27}, \tfrac{17}{27}, \tfrac{19}{27}, \tfrac{20}{27}, \tfrac{22}{27}, \tfrac{23}{27}, \tfrac{25}{27}, \tfrac{26}{27}\right\},\\ \boldsymbol{\beta} &= \left\{\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{2}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \tfrac{5}{6}, \tfrac{1}{7}, \tfrac{2}{7}, \tfrac{3}{7}, \tfrac{4}{7}, \tfrac{5}{7}, \tfrac{6}{7}, 1, 1, 1\right\},\\ t &= 2^{-6}3^{-24}5^{-5}7^{-7}\psi^{-27}.\end{aligned}$$

Theorem

Let $p > 2$ be prime and $q = p^r$ for some $r \in \mathbb{N}_{>0}$,

$$\#X_\psi(\mathbb{F}_q) = q^2 + 2q + 1 + 2q\delta[q \equiv 1 \pmod 3] + H_q(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t).$$

Sketch of Proof

We use the formula of Koblitz for the number of points on a projective hypersurface to write $\#X_\psi(\mathbb{F}_q)$ as a sum involving Gauss sums and characters. Once this is done, we recognise the hypergeometric function.

The L-function

We plug the formula

$$\#X_\psi(\mathbb{F}_q) = q^2 + 2q + 1 + 2q\delta[q \equiv 1 \pmod 3] + H_q(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t)$$

into the series for the zeta function

$$Z_q(X_\psi, T) = \exp\left(\sum_{m=1}^{\infty} \#X_\psi(\mathbb{F}_{q^m}) \frac{T^m}{m}\right)$$

and, after some manipulations, get

Theorem

The (incomplete) L-function of X_ψ factorizes as

$$L_S(X_\psi, s) = \zeta_{S,\mathbb{Q}(i\sqrt{3})}(s-1)\zeta_S(s-1) \cdot L_S\left(H(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t), s\right),$$

where $S = S(\psi)$ is the set of bad primes of X_ψ.

Concluding remarks, References and thank you!

- A similar approach works for the other four pencils.
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