A hypergeometric family of quartics in $\mathbb{P}^3$, periods, point counts and L-function

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(with A. Salerno, E. Orvis, J. Dukes, L. Sturman, R. Davis, U. Whitcher)

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Periods

Definition

Let $X/\mathbb{C}$ be a degree $n+1$ hypersurface in $\mathbb{P}^n$, $\gamma \in H_{n-1}(X,\mathbb{C})$ and $\omega \in H^{n-1}(X,\mathbb{C})$. A *period* of X is an integral

$$\int_\gamma \omega.$$

If X_ψ is a one-parameter family of degree $n+1$ hypersurfaces in $\mathbb{P}^n$, its periods

- are functions $\psi \mapsto \int_{\gamma_\psi} \omega_\psi$.
- satisfy differential equations on ψ, called *Picard-Fuchs* equations.

Example

Example

The Legendre family of elliptic curves

$$C_\lambda : y^2 z = x(x-z)(x-\lambda z), \quad \lambda \in \mathbb{C}$$

has two independent periods with Picard-Fuchs equation given by the Gauss hypergeometric equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\lambda^2} + \frac{1-2\lambda}{\lambda(1-\lambda)}\frac{\mathrm{d}f}{\mathrm{d}\lambda} - \frac{1}{4\lambda(1-\lambda)} f = 0.$$

Near the regular singularity $\lambda = 0$, one of the periods is expressed as the Gauss hypergeometric series

$${}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1, 1 \mid \lambda\right).$$

The relation between periods and point counts

Example

In 1958, Igusa noticed an intriguing relation between the periods and the point-counts of the Legendre family.

Let $\#C_\lambda(\mathbb{F}_p)$ denote the number of points of C_λ over $\mathbb{F}_p$. He showed that for a prime p, the trace of Frobenius

$$a_p(C_\lambda) := p+1-\#C_\lambda(\mathbb{F}_p)$$

satisfies

$$a_p(C_\lambda) \equiv (-1)^{\frac{p-1}{2}} {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1,1 \mid \lambda\right) \pmod{p}.$$

In 1998, Ono showed that using the "finite field version" of ${}_2F_1$, we have the formula

$$\#C_\lambda(\mathbb{F}_q) = q+1-(-1)^{\frac{q-1}{2}} H_q(\tfrac{1}{2},\tfrac{1}{2};1,1 \mid \lambda),$$

where $q = p^r$.

The relation between periods and point counts

Afterwards, the relationship between periods and point counts was explored by more people. A few examples are:

- Pencils of Quintics in $\mathbb{P}^4$ [COR00],[COR04].
- Pencils of Quartics in $\mathbb{P}^3$ [Dor+20].

In this talk,

- another family where the periods and point counts are related by hypergeometric functions.
- the L-functions of this family and its factorisation in terms of hypergeometric L-functions.

Invertible Pencils

If $A = (a_{ij})_{i,j}$ is a $(n+1) \times (n+1)$ matrix of natural numbers. Define:

$$F_A := \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}} \in \mathbb{Z}[x_0, \dots, x_n].$$

A *homogeneous invertible polynomial* is a polynomial of the form F_A satisfying:

1. $\det(A) \neq 0$.
2. F_A is homogeneous of degree $n+1$.
3. $F_A : \mathbb{C}^{n+1} \to \mathbb{C}$ has a unique critical point at the origin.

If F_A is homogeneous invertible, its *transpose* F_{A^T} is quasihomogeneous and defines a hypersurface in $\mathbb{WP}^n(q_0, \dots, q_n)$. We define the *invertible pencil*

$$Y_{A,\psi} := V(F_A - d^T \psi x_0 \cdots x_n) \subset \mathbb{P}^n, \text{ where } d^T = q_0 + \cdots + q_n.$$

Classification

There is a classification of invertible pencils due to Kreuzer and Skarke [KS92]. For $n = 3$, it is

Pencil
$x^4 + y^4 + z^4 + w^4 - 4\psi xyzw$
$x^3y + y^3z + z^3x + w^4 - 4\psi xyzw$
$x^3y + y^3x + z^4 + w^4 - 4\psi xyzw$
$x^3y + y^3x + z^3w + w^3z - 4\psi xyzw$
$x^3y + y^3z + z^3w + w^3x - 4\psi xyzw$
$x^3y + y^4 + z^4 + w^4 - 12\psi xyzw$
$x^3y + y^4 + z^3w + w^3z - 12\psi xyzw$
$x^3y + y^4 + z^3w + w^4 - 6\psi xyzw$
$x^3y + y^3z + z^4 + w^4 - 36\psi xyzw$
$x^3y + y^3z + z^3w + w^4 - 27\psi xyzw$

- (pink) [Dor+20].
- (yellow) [Dav+25]. Today: only the last one.

Hypergeometrics over $\mathbb{C}$

If $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_d\}, \boldsymbol{\beta} = \{\beta_1, \dots, \beta_d\} \subset \mathbb{Q}$ are d-multisets, the *generalised hypergeometric series* associated to $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is defined as

$$ {}_dF_{d-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_d)_n} x^n, $$

where $(a)_n = \begin{cases} a(a+1)\cdots(a+n-1) \text{ if } n > 0 \\ 1, \text{ if } n = 0 \end{cases}.$

If $1 \in \boldsymbol{\beta}$, then ${}_dF_{d-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid x)$ converges for $|x| < 1$ and satisfies

$$ (x\tfrac{\mathrm{d}}{\mathrm{d}x} + \beta_1 - 1)\cdots(x\tfrac{\mathrm{d}}{\mathrm{d}x} + \beta_d - 1)f = x(x\tfrac{\mathrm{d}}{\mathrm{d}x} + \alpha_1)\cdots(x\tfrac{\mathrm{d}}{\mathrm{d}x} + \alpha_d)f. $$

Hypergeometrics over $\mathbb{F}_q$

Let $q = p^r$ and denote $q^\times := q - 1$.

- We would like to define a $\mathbb{F}_q$-version of the hypergeometric series.
- We start by rewriting it in terms of the Γ function

$$ {}_dF_{d-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid z) = \sum_{n=0}^{\infty} \prod_{i=1}^{d} \frac{\Gamma(\alpha_i + n)\Gamma(1 - n - \beta_i)}{\Gamma(\alpha_i)\Gamma(1 - \beta_i)} (-1)^{nd} z^n. $$

- Look for an analogue of the Γ function so that we can imitate the formula above.
- We take ω a generator of the group $\mathrm{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$, Θ non-trivial element of $\mathrm{Hom}((\mathbb{F}_q, +), \mathbb{C}^\times)$, $m \in \mathbb{Z}$ and define the *Gauss sum* as

$$ g(m) = \sum_{x \in \mathbb{F}_q^\times} \omega^m(x)\Theta(x) \in \mathbb{C}. $$

$$ {}_dF_{d-1}(\boldsymbol{\alpha},\boldsymbol{\beta} \mid z) = \sum_{n=0}^{\infty} \prod_{i=1}^{d} \frac{\Gamma(\alpha_i + n)\Gamma(1-n-\beta_i)}{\Gamma(\alpha_i)\Gamma(1-\beta_i)} (-1)^{nd} z^n. $$

Definition (McCarthy,Katz,Greene)

Now suppose that $\boldsymbol{\alpha},\boldsymbol{\beta}$ satisfy $q^\times \alpha_i, q^\times \beta_i \in \mathbb{Z}$, $i = 1,\ldots,d$. For $t \in \mathbb{F}_q^\times$ we define

$$ H_q(\boldsymbol{\alpha},\boldsymbol{\beta} \mid t) = -\frac{1}{q^\times} \sum_{m=0}^{q-2} \prod_{i=1}^{d} \frac{g(m+\alpha_i q^\times)g(-m-\beta_i q^\times)}{g(\alpha_i q^\times)g(-\beta_i q^\times)} \omega((-1)^d t)^m. $$

$$ {}_dF_{d-1}(\boldsymbol{\alpha},\boldsymbol{\beta}\mid z)=\sum_{n=0}^{\infty}\prod_{i=1}^{d}\frac{\Gamma(\alpha_i+n)\Gamma(1-n-\beta_i)}{\Gamma(\alpha_i)\Gamma(1-\beta_i)}(-1)^{nd}z^n. $$

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Warning!

The conditions $q^\times\alpha_i, q^\times\beta_i\in\mathbb{Z}$ for every $i=1,\dots,d$ are restrictive. There is a way to remove them and give a more general definition. I won't talk about it.

Let

$$X_\psi : x^3y + y^3z + z^3w + w^4 - 27\psi xyzw = 0$$

and consider the parameters

$$\boldsymbol{\alpha} = \left\{\tfrac{1}{27}, \tfrac{2}{27}, \tfrac{4}{27}, \tfrac{5}{27}, \tfrac{7}{27}, \tfrac{8}{27}, \tfrac{10}{27}, \tfrac{11}{27}, \tfrac{13}{27}, \tfrac{14}{27}, \tfrac{16}{27}, \tfrac{17}{27}, \tfrac{19}{27}, \tfrac{20}{27}, \tfrac{22}{27}, \tfrac{23}{27}, \tfrac{25}{27}, \tfrac{26}{27}\right\},$$
$$\boldsymbol{\beta} = \left\{\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{2}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \tfrac{5}{6}, \tfrac{1}{7}, \tfrac{2}{7}, \tfrac{3}{7}, \tfrac{4}{7}, \tfrac{5}{7}, \tfrac{6}{7}, 1, 1, 1\right\},$$
$$t = 2^{-6}3^{-24}5^{-5}7^{-7}\psi^{-27}.$$

Theorem

$H^2_{prim}(X_\psi, \mathbb{C})$ has 18 periods associated to the hypergeometric funtion ${}_{18}F_{17}(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t)$.

Theorem

Let $p > 2$ be prime and $q = p^r$ for some $r \in \mathbb{N}_{>0}$,

$$\#X_\psi(\mathbb{F}_q) = q^2 + 2q + 1 + 2q\delta[q \equiv 1 \pmod 3] + H_q(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t).$$

The L-function as a by-product (of the finite field result)

The advantage of having the point counts written as hypergeometrics is that we can now easily find the L-function and its factorization.

Corollary

The (incomplete) L-function of X_ψ factorizes as

$$L_S(X_\psi, s) = \zeta_{S,\mathbb{Q}(i\sqrt{3})}(s-1)\zeta_S(s-1) \cdot L_S(H(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t), s),$$

where $S = S(\psi)$ is the set of bad primes of X_ψ.

References

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Thank you!