Hypergeometric structures for invertible pencils of quartics in $\mathbb{P}^3$

Work in progress with Adriana Salerno, Eli Orvis, Jessamyn Dukes, Leah Sturman, Rachel Davis and Ursula Whitcher

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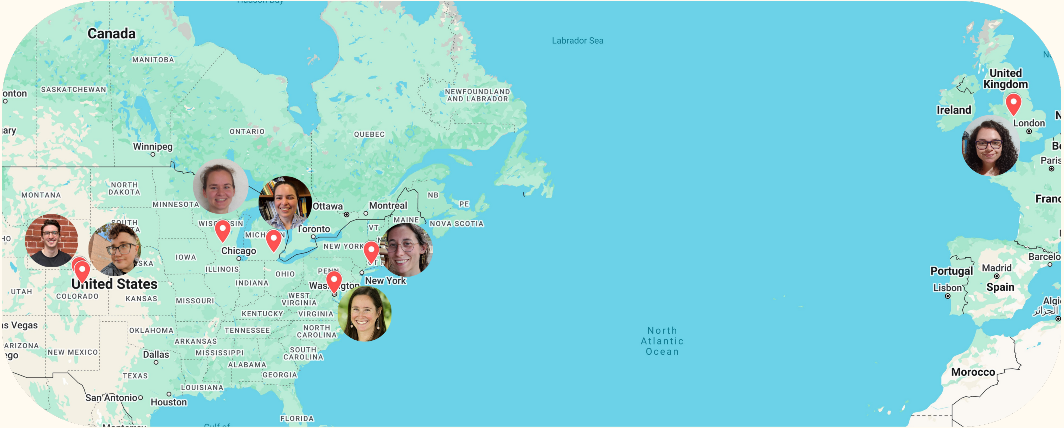
Our group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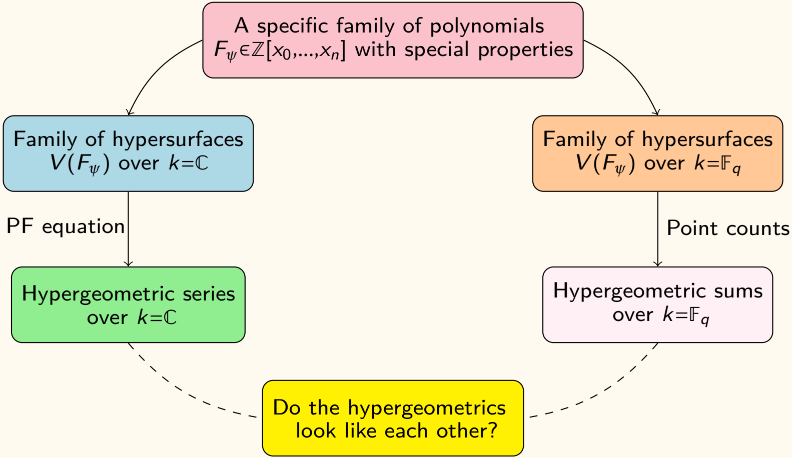
Goal of the talk



Invertible polynomials over an arbitrary field k

Definition

For any field k, if $A = (a_{ij})_{i,j}$ is a $(n+1)\times(n+1)$ matrix with natural numbers in all the entries, we can define:

$$F_A := \sum_{i=0}^{n}\prod_{j=0}^{n} x_j^{a_{ij}} \in \mathbb{Z}[x_0,\ldots,x_n].$$

We say that F_A is invertible if:

1. A is invertible.
2. F_A is homogeneous of degree $n+1$.

Example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \begin{matrix} (1\ \ 1) \longrightarrow X_0^1X_1^1 \\ (0\ \ 2) \longrightarrow X_0^0X_1^2 \end{matrix} \qquad F_A = X_0X_1 + X_1^2$$

Transposed polynomial (over an arbitrary field k)

F_A invertible

↓

F_{A^T} **quasihomogeneous**

↓

hypersurface in $\mathbb{WP}^n(\underbrace{q_0,\dots,q_n}_{\text{dual weights}})$

↓

$d^T := q_0 + \cdots + q_n$

Deformations

Definition (*Invertible pencil*)

If k is any field and F_A is invertible, we define the one-parameter deformation

$$Y_{A,\psi} := V(F_A - d^T \psi x_0 \cdots x_n) \subset \mathbb{P}^n_k.$$

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Example

Consider the family of surfaces in $\mathbb{P}^3_k$ given by:

$$Y_\psi : x^3 y + y^4 + z^3 w + w^3 z - 12\psi xyzw = 0.$$

$n = 3$, variables x, y, z, w and $k = \mathbb{C}$

Pencil	dual weights
$x^4 + y^4 + z^4 + w^4 - 4\psi xyzw$	(1,1,1,1)
$x^3y + y^3z + z^3x + w^4 - 4\psi xyzw$	(1,1,1,1)
$x^3y + y^3x + z^4 + w^4 - 4\psi xyzw$	(1,1,1,1)
$x^3y + y^3x + z^3w + w^3z - 4\psi xyzw$	(1,1,1,1)
$x^3y + y^3z + z^3w + w^3x - 4\psi xyzw$	(1,1,1,1)
$x^3y + y^4 + z^4 + w^4 - 12\psi xyzw$	(4,2,3,3)
$x^3y + y^4 + z^3w + w^3z - 12\psi xyzw$	(4,2,3,3)
$x^3y + y^4 + z^3w + w^4 - 6\psi xyzw$	(2,1,2,1)
$x^3y + y^3z + z^4 + w^4 - 36\psi xyzw$	(12,8,7,9)
$x^3y + y^3z + z^3w + w^4 - 27\psi xyzw$	(9,6,7,5)

Studied in [Doran et al., 2020] by Adriana, Ursula and their collaborators. (pink rows)

Ongoing work. (yellow rows)

Why are we interested in those pencils?

- Related to BHK mirror symmetry;
- In previous work ([Doran et al., 2020]), Adriana, Ursula and their collaborators showed that for 5 of those pencils there is a way to decompose the middle cohomology via hypergeometric differential operators;
- This decomposition is related to L-series;
- Smooth members of the families are K3 surfaces;

Hypergeometric series and differential equations ($k = \mathbb{C}$)

Given a family X_ψ of projective hypersurfaces in $\mathbb{P}^n$, we can associate to it some differential equations on the parameter ψ that are called the Picard-Fuchs equations of the family.

The Picard-Fuchs equation associated with the holomorphic form of the family

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Definition

The Pochhammer symbol is defined by $(a)_n = a(a+1)\cdots(a+n-1)$, for $a \in \mathbb{Q}$ and $n > 1$ natural and $(a)_0 = 1$.

Hypergeometric differential equation

Definition

Let $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_d\}, \boldsymbol{\beta} = \{\beta_1, \dots, \beta_d\} \subset \mathbb{Q}$ be d-multisets (sets where elements can repeat). The *hypergeometric series* associated to $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is defined as

$$ {}_dF_{d-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} | x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_d)_n} x^n. $$

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Proposition

If $\beta_1 = 1$, then $D(\boldsymbol{\alpha}, \boldsymbol{\beta}|x) \cdot {}_dF_{d-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}|x) = 0$.

Changing to arithmetic ($k = \mathbb{F}_q$)

Let's pause and change the scenario a bit to explore the case $k = \mathbb{F}_q$. Consider the same family, but in $\mathbb{P}^3_{\mathbb{F}_q}$:

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- Denote $q^\times := q - 1$.
- ω - generator of the group $\mathrm{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$.
- Θ non-trivial element of $\mathrm{Hom}((\mathbb{F}_q, +), \mathbb{C}^\times)$.

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Definition

Given $m \in \mathbb{Z}$, we define the *Gauss sum* to be

$$g(m) = \sum_{x \in \mathbb{F}_q^\times} \omega^m(x)\Theta(x) \in \mathbb{C}^\times.$$

Field of definition

$\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_d\}$ and $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_d\} \rightsquigarrow$ multisets in $\mathbb{Q}$

$$\boldsymbol{\alpha} \rightsquigarrow F(x) := \prod_{i=1}^{d} (x - e^{2\pi i \alpha_i}) \qquad \boldsymbol{\beta} \rightsquigarrow G(x) := \prod_{i=1}^{d} (x - e^{2\pi i \beta_i})$$

$K_{\boldsymbol{\alpha},\boldsymbol{\beta}} \rightsquigarrow$ field generated over $\mathbb{Q}$ by the coefficients of F and G

Definition

If $K_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathbb{Q}$, we say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are defined over $\mathbb{Q}$.

Ingredients to produce a finite field hypergeometric sum

Suppose that $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are defined over $\mathbb{Q}$ and that $\alpha_i - \beta_j \notin \mathbb{Z}$ for every i, j. We write

$$\frac{F}{G} = \frac{\prod_{j=1}^{r} x^{p_j} - 1}{\prod_{j=1}^{s} x^{q_j} - 1} \quad \text{where} \quad \{p_1, \ldots, p_r\} \cap \{q_1, \ldots, q_s\} = \varnothing.$$

$$D(x) = \gcd\left(\prod_{j=1}^{r} x^{p_j} - 1, \prod_{j=1}^{s} x^{q_j} - 1\right), \epsilon = (-1)^{\sum_{j=1}^{s} q_j}, M = \prod_{j=1}^{r} p_j^{p_j} \prod_{j=1}^{s} q_j^{-q_j}.$$

$$s : \mathbb{Z}/q^{\times}\mathbb{Z} \to \mathbb{Z} \text{ defined by } m \mapsto \text{multiplicity of } e^{\frac{2\pi i m}{q^{\times}}} \text{ in } D(x).$$

Definition

Suppose that $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are defined over $\mathbb{Q}$. Given $t \in \mathbb{F}_q^{\times}$, define the finite field hypergeometric sum

$$H_q(\boldsymbol{\alpha}, \boldsymbol{\beta} | t) = \frac{(-1)^{r+s}}{1-q} \sum_{m=0}^{q-2} q^{-s(0)+s(m)} \prod_{i=1}^{r} g(p_i m) \prod_{i=1}^{s} g(-q_i m) \omega(\epsilon M^{-1} t)^m.$$

Hypergeometric Picard-Fuchs vs hypergeometric point count of $Y_\psi : x^3y+y^4+z^3w+zw^3-12\psi xyzw=0$ in $\mathbb{P}^3_k$

Proposition

If $k=\mathbb{C}$, then the Picard-Fuchs equation satisfied by the homolomorphic form $\operatorname{Res}\left(\frac{1}{F_\psi}\Omega_0\right)$ of Y_ψ is the hypergeometric differential equation given by the operator

$$D\left(\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12},\frac{1}{6},\frac{5}{6};1,1,1,\frac{1}{2},\frac{1}{3},\frac{2}{3}\,\middle|\,2^{-10}3^{-6}\psi^{-12}\right).$$

If $k=\mathbb{F}_q$, then

$$\begin{aligned}\#Y_\psi(\mathbb{F}_q) &= q^2+2q+1+4q\delta[q\equiv 1 \pmod 3]+2q\delta[q\equiv 1 \pmod 4]\\ &+H_q\left(\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12},\frac{1}{6},\frac{5}{6};1,1,1,\frac{1}{2},\frac{1}{3},\frac{2}{3}\,\middle|\,2^{-10}3^{-6}\psi^{-12}\right)\\ &+\left(\frac{-12\psi}{q}\right)qH_q\left(\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24};0,\frac{1}{6},\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{5}{6}\,\middle|\,2^{-10}3^{-6}\psi^{-12}\right).\end{aligned}$$

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It also has a correspondent in the complex analytic side!

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Proposition

The Picard-Fuchs satisfied by the form $\mathrm{Res}\left(\frac{xy^2w}{F_\psi}\Omega_0\right)$ is hypergeometric and has as one of its solutions the hypergeometric series

$$\psi \, {}_8F_7\left(\tfrac{1}{24},\tfrac{5}{24},\tfrac{7}{24},\tfrac{11}{24},\tfrac{13}{24},\tfrac{17}{24},\tfrac{19}{24},\tfrac{23}{24};0,\tfrac{1}{6},\tfrac{1}{4},\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3},\tfrac{3}{4},\tfrac{5}{6} \mid \psi^{-24}\right). \tag{4}$$

Proof.

The proof follows from applying a more general theorem proved in [Adolphson and Sperber, 2023]. □

Summary

Yes, they look like each other. In one side, if you input $\boldsymbol{\alpha}, \boldsymbol{\beta}, t$ you solve differential equations, and on the other side, you count points.

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References

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Thank you!